Chapter 1

Definition of probability.

In both nature and the various events that happen in our lives, there are phenomena that occur and repeat, which are surrounded by uncertainty and unknowns. For example, let's consider a certain industrial process that produces a certain item in large quantities. Obviously, not all items are identical; each manufactured item may turn out to be good or defective without us being able to predict it in advance. This is a random situation that can arise in real life. In these types of situations, we make use of Probability Calculus because we need to somehow measure the chances of the item being defective or not. Probability is the likelihood of something happening. What we need is a tool that provides us with a measure of uncertainty about the realization of any random event.

1.1 Axiomatic definition.

This definition was given by the Russian mathematician Andrei Kolmogorov and is the basis for the modern development of Probability Calculus.

Definition. Let $(\Omega, \mathcal{A})$ be A measurable space associated with a random experiment. Then the function

$$p : \mathcal{A} \Longrightarrow \Re$$

is probability on $(\Omega, \mathcal{A})$ if it verifies:

1. $p(A) \geq 0$, for any event $A \in \mathcal{A}$.
2. $p(\Omega) = 1$.
3. If $\{A_1, A_2, \ldots, A_n\} \in \mathcal{A}, \{A_i \cap A_j \neq 0\}$, then $p(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} p(A_i)$.

This axiomatic definition can be somewhat abstract, since it does not tell us at any moment what probability to assign to any particular event. Let's see how to deduce this from the Definition itself.

Let's suppose that the sample space of a random experiment is composed of n possible outcomes, that is, of n elementary events of the form $\Omega = w_1, w_2, \ldots, w_n$. If we have an event A, it will be formed by a certain number of elementary events $w_1, w_2, \ldots, w_k$, so we can write: $A = w_1, w_2, \ldots, w_k = w_1 \cup w_2 \cup \ldots \cup w_k$, this being a disjoint union, since the elementary events are incompatible with each other as they are different results of an experiment. Applying axiom 3:

$$p(A) = \sum_{i=1}^{k} p(w_i).$$

Esto nos dice que para calcular la probabilidad de cualquier suceso A asociado a un experimento aleatorio, debemos conocer de antemano la probabilidad de ocurrencia de todos los resultados posibles de ese experimento. Estas probabilidades se suelen llamar *pesos.*

Equally likely elementary events

A particular case is when we assume that all possible outcomes of a random experiment are equiprobable, for example, in the toss of an unbiased coin or a fair die.

Let's imagine that the experiment has n equiprobable possible outcomes, that is, $p(w_i) = q$. Then,

$$p(\Omega) = \sum_{i=1}^{n} p(w_i) = q\dot{n} = 1, \qquad \text{luego} \quad q = 1/n$$

If we have an event A formed by a certain number of elementary events $w_1, w_2, \ldots, w_k$, the probability of A will be:

$$p(A) = \sum_{i=1}^{k} \frac{1}{n} = \frac{k}{n}$$

Therefore, we can say that the probability of an event A is the ratio between the number of favorable outcomes to event A divided by the total number of possible outcomes of the experiment. This is known as the *Rule of Laplace.*

1.2 Conditional probability.

Definition. Given the events A and B, we call the *probability of B given A* to the number represented by $P(B/A)$ and calculated as:

$$P(B/A) = \frac{p(A \cap B)}{p(A)}$$

Compound probability

We talk about *compound probability* as the probability of two or more events occurring simultaneously. From the Definition of conditional probability, it follows that:

$$p(A \cap B) = p(A) \cdot p(B/A)$$

Remark

Since the intersection operation of events has the commutative property, i.e., $A \cap B = B \cap A$, we can write:

$$p(A \cap B) = p(B) \cdot p(A/B)$$

Example

In a certain school, 50% of the students speak English. Within the group of those who speak English, 20% speak German. Calculate the probability that, chosen a student at random, they speak both languages. We have the following events:

"A: "The student speaks English" and B: "The student speaks German".

We have been provided with the following data:

$p(A) = 0,5$ y $p(B/A) = 0,2$.

This last piece of information is sometimes confused with the intersection of events A and B. But it is a conditional probability since they tell us "within the group of English speakers", meaning event A has already happened once we have chosen a student from that group.

$$p(A \cap B) = p(A) \cdot p(B/A) = 0,5 \cdot 0,2 = 0,1$$

Independent events

When events A and B are independent, the fact that one of them has occurred previously does not affect the probability of the other event happening, that is, $p(A/B) = p(A)$ and $p(B/A) = p(B)$. Therefore,

- $p(A \cap B) = p(B) \cdot p(A/B) = p(B) \cdot p(A) = p(A) \cdot p(B)$
- $p(A \cap B) = p(A) \cdot p(B/A) = p(A) \cdot p(B)$

We can say that two events A and B are independent if it is verified that:

$$p(A \cap B) = p(A) \cdot p(B)$$

Note. We can expand the formula for compound probability to any number of events. For example, for events A, B, and C, we would have:

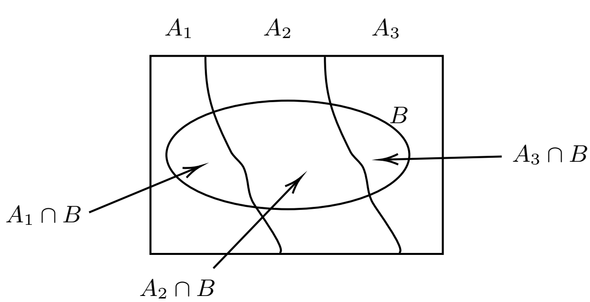
$$p(A \cap B \cap C) = p(A) \cdot p(B/A) \cdot p(C/A \cap B)$$

1.2.1 Law of Total Probability

If we have any event B, we can express it in terms of another set of events $A_1, A_2, \ldots, A_n$ that form a *complete system*, that is, the union of all the events is equal to the sample space Ω, and the events are pairwise disjoint. If we simplify this system to three events A_1, A_2, A_3 -which will be usual in the problems we solve-, we can express event B as: $B = (A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B)$ (see diagram), then:

$$\begin{aligned} p(B) &= p(A_1 \cap B) + p(A_2 \cap B) + p(A_3 \cap B) = \\ &= p(A_1) \cdot (B/A_1) + P(A_2) \cdot p(B/A_2) + p(A_3) \cdot p(B/A_3) \end{aligned}$$

This is known as the *Law of Total Probability.*



Example

A car factory produces annually 2000 SUV vehicles, 3000 sedans, and 5000 station wagons. It has been proven that 1% of SUVs have issues with starting, the same happening with 2% of sedans and 3% of station wagons. Calculate the probability that, chosen a car at random, it presents issues with starting.

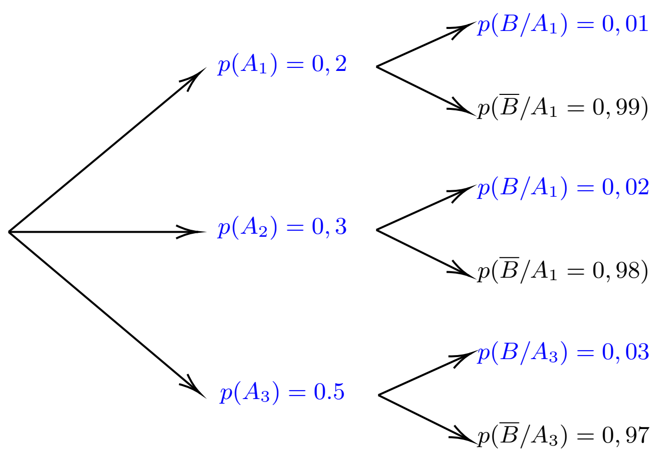
Let us define the following events:

- A_1: "The produced car is an SUV".
- A_2: "The produced car is a sedan".
- A_3: "The produced car is a station wagon".
- B: "The vehicle has issues with starting".

then:

$p(B) = p(A_1)p(B/A_1) + P(A_2)p(B/A_2) + p(A_3)p(B/A_3) = 0,2 \cdot 0,01 + 0,3 \cdot 0,02 + 0,5 \cdot 0,03 = 0,023$

We can solve this type of problems using a tree diagram:



To calculate the probability of event B, we have to select the branches where this event occurs, multiply the elements of each branch, and add everything up.

1.2.2 Bayes' Theorem

This famous theorem is used to modify our subjective probabilities (*a priori*) when we receive additional information.

Example

A medical team establishes that the probability of having hepatitis in a certain population is 0.01%. We call this probability subjective probability (*a priori*). Through daily experience, it is known that the probability of a hepatitis detection test showing a positive result in sick people is 99%, while in healthy people the test is positive in 0.5% of cases. The usefulness of this theorem is that it allows us to update the prior probability if we have additional information, that is:

- A: "The individual has hepatitis".
- B: "The test has shown a positive result for hepatitis".

$$p(A/B) = \frac{p(A) \cdot p(B/A)}{p(B)}$$

What the medical team wants is to calculate the incidence of hepatitis in the population as reliably as possible. Since they now have the additional information provided by the detection tests, they will update the probability of event A, that is, they will calculate $p(A/B)$ (posterior probability). The probability $p(B/A)$ is called *likelihood*. It is this likelihood that updates the probability of A. To calculate $p(B)$, we use the total probability.

$$p(A/B) = \frac{0,0001 \cdot 0,99}{0,0001 \cdot 0,99 + 0,9999 \cdot 0,005} = 0,0194$$

With this new data, they can conclude that the incidence of hepatitis in the population is 1.94%.

1.2.3 Summary of basic properties:

In this section, we will summarize the formulas, theorems, and properties we will use to solve probability problems:

- $p(\overline{A}) = 1 - p(A)$
- Fórmula general: $p(A \cup B) = p(A) + p(B) - p(A \cap B)$
- $p(A \cup B) = p(A) + p(B)$, si A y B son incompatibles ($p(A \cap B) = 0$)
- $p(A \cap \overline{B}) = p(A) - p(A \cap B)$
- $p(\overline{A} \cap B) = p(B) - p(A \cap B)$
- De Morgan's Laws:

1. $p(\overline{A} \cap \overline{B}) = p(\overline{A \cup B}) = 1 - p(A \cup B)$
2. $p(\overline{A} \cup \overline{B}) = p(\overline{A \cap B}) = 1 - p(A \cap B)$

- $p(A \cap B) = p(A) \cdot p(B/A)$
- $p(A \cap B) = p(A) \cdot p(B)$, if A and B are independent events.
- $p(A/B) = \dfrac{p(A \cap B)}{p(B)}$
- Theorem of Total Probability: $p(A) = p(A_1) \cdot p(A/A_1) + \ldots + p(A_n) \cdot p(A/A_n)$
- Bayes' Theorem: $p(A_i/A) = \dfrac{p(A_i) \cdot p(A/A_i)}{A}$

1.3 Basic notions of combinatorics

Variation withouth repetition

Let U be a set of N elements and let n be any natural number, $n \leq N$, We define the *variation without repetition* of n elements as the different groups of n elements that can be done, so that two groups differ from each other only in the order the elements are placed. Let us see how many permutations can be counted:

For i_1 there will be N cases, for i_2 N-1 and so on. Then:

$$|\Omega| = N(N-1)(N-2).....(N-n+1) \equiv V_{N,n}$$

Example. In the football league, 20 teams participate. How many possible ways can the top three places in the championship be determined?

We have 20 elements and we group them in threes. Order matters since it's not the same to finish first as it is to finish third. Therefore,

$$V_{20,3} = 20 \cdot 19 \cdot 18 = 6840$$

Variation with repetition

In this case, order still matters but elements can be repeated. For i_1 there will be N possibilities, for i_2 we will have N possibilities, and so on:

$$|\Omega| = N \cdot N \cdots N = N^n \equiv VR_{N,n}$$

Example

The Quiniela de fútbol consists of predicting the outcome of 15 football matches. Each match has three possible options: home team victory, draw or away team victory. How many bets can we make?

he total number of possible outcomes is 3 to the power of 15 (since there are 15 matches),

$$VR_{3,15} = 3^{15}$$

Permutation

Permutations of order N are a particular case of variations without repetition when we have N elements in a set and make variations of order N:

$$P_N = N! = N(N-1).....(N-N+1) = N(N-1)(N-2)\dots 1 = V_{N,N}$$

$N!$ is called the factorial of N.

Permutation with repetition

Let's suppose we have a set of N elements that are grouped into k groups of equal elements, that is: $N = N_1 + N_2 + ... + N_k$. Permutations with repetition are the permutations that can be performed with N elements ($N!$), but with the exception that, in this case, and because we have k groups of equal elements, we don't have to take into account the order between equal elements. Therefore, the number of permutations with repetition will be $N!$ divided by the product of all the permutations that can be made within each group:

$$P_N^{N_1,\dots,N_k} = \frac{N!}{N_1!N_2!...N_k!}$$

Example

How many groupings can be formed with the word PATATA? We will have N=6 elements grouped into three subsets of 3, 2, and 1 element (letters A, T, and P, respectively). Then:

$$P_6^{3,2,1} = \frac{6!}{3!\cdot 2!\cdot 1!}$$

Combination.

A different way of counting than the previous ones is through combinations. The difference with respect to variaations lies in the fact that now *the order does not matter.*

The difference between permutations and combinations is that, in permutations, the position of the elements *does matter*. Since each subset of n elements can be arranged in $n!$ different ways, it is clear that the number of combinations of N elements taken n at a time, $C_{N,n}$, will be:

$$|\Omega| = C_{N,n} = \frac{V_{N,n}}{n!}$$

Remark

$$C_{N,n} = \binom{N}{n} = \frac{N!}{(N-n)!n!}$$

Example

From a syllabus of 70 topics, in how many possible ways can I choose two topics?

It is clear that choosing {topic1, topic2} is the same as choosing {topic2, topic1}, so the order doesn't matter. We will then calculate the number of subsets of two topics that we can form from 70. Thus, we have:

$$C_{70,2} = \binom{70}{2} = \frac{70!}{(70-2)!2!} = 2415$$

Combination with repetition

Now we will consider all the subsets of n elements, allowing for repeated elements. The formula is as follows:

$$|\Omega| = CR_{N,n} = \binom{N+n-1}{n}$$

1.4 Excercises

Excercise 1.

How many ways can 10 people sit in a line if there are 4 available seats?

Solution. EIn this case we will use variations because order matters. Since a person cannot occupy more than one seat at a time, we will use variations without repetition:

$$V_{10,4} = \frac{10!}{(10-7)!} = 5040$$

Excercise 2.

In a class of 10 students, 3 prizes will be awarded. In how many ways can this be done if:

1. The prizes are different.
2. The prizes are the same.

Solution. Assuming that a person cannot receive more than one prize,

1. Different prizes (order matters): $V_{10,3} = 720$
2. Same prizes (order does not matter): $C_{10,3}$Ç $= 120$

Excercise 3.

We need to place five men and four women in a line so that the women occupy the even positions. How many ways can this be done? **Solution.**

$$\underline{\ H_1\ }\ \underline{\ M_1\ }\ \underline{\ H_2\ }\ \underline{\ M_2\ }\ \underline{\ H_3\ }\ \underline{\ M_3\ }\ \underline{\ H_4\ }\ \underline{\ M_3\ }\ \underline{\ H_5\ }$$

Hay P_5 formas de sentar a los hombres, mientras que a las mujeres las sentaremos de P_4 maneras diferentes. Con lo cual, el número total será

$$P_5 \cdot P_4 = 5! \cdot 4! = 2880$$

Excercise 4.

How many four-digit numbers can be formed using the digits $0, 1, \ldots, 9$ without repetition, and with the last digit being zero? **Solution.**

$$\underbrace{\underline{\quad}\ \underline{\quad}\ \underline{\quad}}_{V_{9,3}}\ \underline{\ 0\ }$$

The digit 0 will be fixed in the last position. Therefore, I can vary the remaining nine digits without repetition, so the number of possible numbers will be:

$$V_{9,3} = 9 \cdot 8 \cdot 7 = 504$$

Excercise 5.

One student has to choose seven out of ten questions in an exam. In how many different ways can they choose them? And what if the first four are mandatory?

Solution. We will use combinations because the order in which the questions are chosen is irrelevant. Since the questions cannot be repeated, we will use combinations without repetition. Therefore, the number of questions will be:

$$C_{10,7} = \frac{10!}{7!(10-7)!} = 120$$

If the first four questions are mandatory, we must choose three questions from the remaining six:

$$C_{6,3} = \frac{6!}{7!(6-3)!} = 20$$

Excercise 6.

Given two random events A and B, it is verified that $p(A) = 2p(B)$. Besides, $p(A) + p(\overline{B}) = 1.3$ and $p(A \cap B) = 0.18$. Calculate the probability that:

1. Verify A or verify B.
2. Verify $\overline{A} \cap \overline{B}$.
3. Are A and B independent?.

Solution. First we have to solve a system of equations to calculate the values of $p(A)$ and $p(B)$:

$$\begin{cases} p(A) = 2p(B) \\ p(A) + p(\overline{B}) = 1,3 \end{cases}$$

If we substitute $p(A) = 2p(B)$ into the second equation, we get $2p(B)+p(\overline{B}) = 1.3$. Since $p(\overline{B}) = 1 - p(B)$, we have $3p(B) = 1.3$ which implies $p(B) = 0.3$ and $p(A) = 2p(B) = 0.6$

1. $p(A \cup B) = p(A) + p(B) - p(A \cap B) = 0.6 + 0.3 - 0.18 = 0.72$
2. $p(\overline{A} \cap \overline{B}) = p(\overline{A \cup B}) = 1 - p(A \cup B) = 1 - 0.72 = 0.28$

3. $p(A) \cdot p(B) = 0.18 = p(A \cap B)$, therefore, they are independent.

Excercise 7.

Given two events A and B such that: $p(A) = \frac{3}{8}$, $p(B) = \frac{1}{2}$ y $p(A \cap B) = \frac{1}{4}$, calculate:

1. $p(A \cup B)$.
2. $p(\overline{A} \cap \overline{B})$.
3. $p(\overline{A} \cup \overline{B})$.
4. $p(A/\overline{B})$.
5. $p(B/\overline{A})$.

Solution.

1. $p(A \cup B) = p(A) + p(B) - p(A \cap B) = \frac{3}{8} + \frac{1}{2} - \frac{1}{4} = \frac{5}{8}$
2. $p(\overline{A} \cap \overline{B}) = p(\overline{A \cup B}) = 1 - p(A \cup B) = 1 - \frac{5}{8} = \frac{3}{8}$
3. $p(\overline{A} \cup \overline{B}) = p(\overline{A \cap B}) = 1 - p(A \cap B) = 1 - \frac{1}{4} = \frac{3}{4}$
4. $p(A/\overline{B}) = \dfrac{p(A \cap \overline{B})}{p(\overline{B})} = \dfrac{p(A) - p(A \cap B)}{p(\overline{B})} = \dfrac{\frac{3}{8} - \frac{1}{4}}{1 - \frac{1}{2}} = \frac{1}{4}$
5. $p(B/\overline{A}) = \dfrac{p(B \cap \overline{A})}{p(\overline{A})} = \dfrac{p(B) - p(A \cap B)}{p(\overline{A})} = \dfrac{\frac{1}{2} - \frac{1}{4}}{1 - \frac{3}{8}} = \frac{2}{5}$

Excercise 8.

In an athletics club, the following holds true:

- 75% of the members participate in mid-distance races.
- 70% of the members participate in long-distance races.

- 13% of the members do not participate in these types of races.

Chosen at random a member of the club, calculate the probability that they participate only in long-distance races or only in medium-distance races. **Solution.** We define the following events:

- L: "Participate in long-distance races".
- M: "Participate in mid-distance races".

We must calculate $p\left[(L\cap\overline{M})\cup(\overline{L}\cap M)\right]$. It is clear that the events L and M are incompatible, therefore:

$$p\left[(L\cap\overline{M})\cup(\overline{L}\cap M)\right]=p((L\cap\overline{M})+p(\overline{L}\cap M)$$

(1)

Applying the properties of events, we have that

$$p(L\cap\overline{M})=p(L)-P(L\cap M)\quad \text{y} \quad p(\overline{L}\cap M)=p(M)-p(L\cap M)$$

We would only have to calculate $p(L\cap M)$. We can write $p(\overline{L}\cap\overline{M})=0.13$. Then,

$p(L\cap M)=p(L)+p(M)-p(L\cup M)=p(L)+p(M)-(1-p(\overline{L}\cap\overline{M}))=0.70+0.75-(1-0.13)=0.58.$

Replacing in (1):

$$p\left[(L\cap\overline{M})\cup(\overline{L}\cap M)\right]=p(L\cap\overline{M})+p(\overline{L}\cap M)=0.7-0.58+0.75-0.58=0.29$$

Excercise 9.

given two events A y B, so that $p(\overline{A})=\dfrac{2}{3}$, $p(A\cup B)=\dfrac{3}{4}$ y $p(A\cap B)=\dfrac{1}{4}$ calculate:

1. $p(A)$
2. $p(B)$
3. $p(A\cap\overline{B})$
4. $p(B\cap\overline{A})$

Solution.

- $p(A) = 1 - p(A) = 1 - \frac{2}{3} = \frac{1}{3}$
- $p(B) = p(A \cup B) - p(A) + p(A \cap B) = \frac{3}{4} - \frac{1}{3} + \frac{1}{4} = \frac{2}{3}$
- $p(A \cap \overline{B}) = p(A) - p(A \cap B) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$
- $p(B \cap \overline{A}) = p(B) - p(A \cap B) = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$

Excercise 10.

The Duke of Tuscany once asked Galileo: Why is it that when three dice are thrown and their points are added, a sum of 10 is obtained more often than a sum of 9, even though they can be obtained in six different ways each?
Solution. Indeed, the 9 and 10 can be obtained in 6 different ways:

$$\text{number 9} \begin{cases} 1,4,4 \\ 1,2,6 \\ 2,5,2 \\ 3,1,5 \\ 3,2,4 \\ 3,3,3 \end{cases} \qquad \text{number 10} \begin{cases} 1,5,4 \\ 1,3,6 \\ 2,2,6 \\ 3,2,5 \\ 2,2,4 \\ 3,4,3 \end{cases}$$

The Duke of Tuscany did not take into account that not all cases are equiprobable because, for example, the outcome $\{1,3,5\}$ can occur in three different ways (by combining the results of the dice), while $\{3,3,3\}$ can only occur in one way. We can see this in the following graph:

$$\text{number 9} \begin{cases} 144 \longrightarrow & 3 \\ 126 \longrightarrow & 6 \\ 252 \longrightarrow & 3 \\ 315 \longrightarrow & 6 \\ 324 \longrightarrow & 6 \\ 333 \longrightarrow & 1 \end{cases} \qquad \text{number 10} \begin{cases} 154 \longrightarrow & 6 \\ 136 \longrightarrow & 6 \\ 226 \longrightarrow & 3 \\ 325 \longrightarrow & 6 \\ 224 \longrightarrow & 3 \\ 343 \longrightarrow & 3 \end{cases}$$

	possibilities		possibilities
total	25	total	27

Therefore, it is more likely to obtain 10 points than 9. Its probability will be:

$$p(\text{“To get 10 points”}) = \frac{27}{6^3} = \frac{1}{8}$$

Excercise 11.

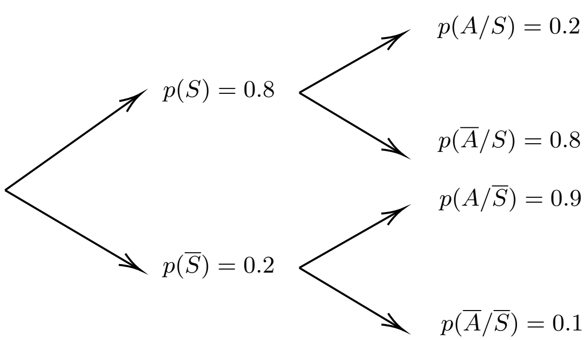
Javier's alarm clock does not work very well, as it does not ring 20% of the time. When it does ring, Javier is late for class with probability 0.2, but if it doesn't ring, the probability of him being late is 0.9.

1. Determine the probability of the alarm clock having gone off and arriving late to class.
2. Determine the probability of arriving early to class on any given day.
3. Javier arrives late to class one day. What is the probability that his alarm clock didn't go off?.

Solution. We define the following events:

- S : “The alarm clock has gone off”.
- A : “Arriving late to class”.

We represent the tree diagram:



1. We always select the branch where both events, in this case S y A: $p(S \cap A) = p(S) \cdot p(A/S) = 0,8 \cdot 0,2 = 0,16$
2. We use the Law of Total Probability. In a less rigorous way, we can say that it is the branch where event $\overline{A}$ appears:
   $p(\overline{A}) = p(S) \cdot p(\overline{A}/S) + p(\overline{S}) \cdot p(\overline{A}/\overline{S}) = 0,8 \cdot 0,8 + 0,2 \cdot 0,1 = 0,66$

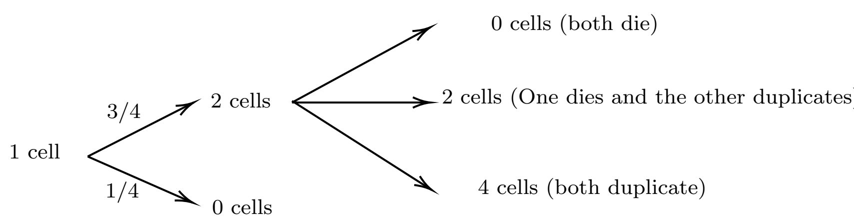
3. We apply Bayes' theorem:

$$p(\overline{S}/A) = \frac{p(\overline{S} \cap A)}{A} = \frac{p(\overline{S}) \cdot p(A/\overline{S})}{1 - p(A)} = \frac{0,2 \cdot 0,9}{1 - 0,66} = 0,529$$

Excercise 12.

Consider a cell at time $t = 0$. At time $t = 1$, the cell can either reproduce, dividing into two cells with a probability of $3/4$, or die with a probability of $1/4$. If the cell divides, then at time $t = 2$, each of its two descendants can also either subdivide or die, with the same probabilities as before, independently of each other. How many cells could there be at time $t = 2$, and with what probabilities?.

Solution. We describe the excercise by representing the following diagram:



As we can see, after the second division, there can only be four, two, or no cells. Let's call N to the number of cells at $t = 2$. Since the divisions are independent of each other, we can multiply the probability values:

$$p(N = 4) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$$

$$p(N = 2) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{9}{32}$$

$$p(N = 4) = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{3}{64}$$

Excercise 13.

Students A and B have probabilities of $1/2$ and $1/5$, respectively, of failing an exam. The probability that they both fail the exam simultaneously is $1/10$. Determine the probability that only one of the two students fails the exam.

Solution. If we call A and B the events "failure of student A" and "failure of student B", respectively, then we will have to calculate:

$$p\left[(A \cap \overline{B}) \cup (\overline{A} \cap B)\right] = p(A \cap \overline{B}) + p(\overline{A} \cap B),$$

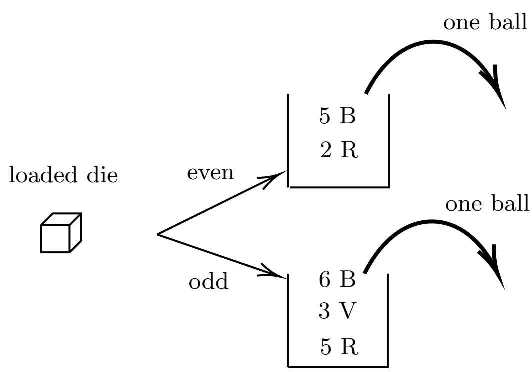
since $(A \cap \overline{B})$ and $(\overline{A} \cap B)$ are mutually exclusive events.

Furthermore, it is verified that $p(A \cap B) = p(A) \cdot p(B)$, so A and B are independent. Therefore,

$p\left[(A \cap \overline{B}) \cup (\overline{A} \cap B)\right] = p(A \cap \overline{B}) + p(\overline{A} \cap B) = p(A) \cdot p(\overline{B}) + p(\overline{A}) \cdot p(B) = \frac{1}{2} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{2}$

Excercise 14.

In urn A, there are 5 white and 2 red balls, and in urn B there are 3 green, 6 white, and 5 red balls. A loaded die is rolled, with faces numbered from 1 to 6, and the probability of getting a six is twice that of getting any other number. If an even number is rolled, a ball is drawn from urn A, and if an odd number is rolled, a ball is drawn from urn B. Determine the probability that the ball drawn is red. **Solution.** We represent the problem in the following diagram:



We must first calculate the probability of obtaining any number in the roll of the loaded die:

We call $p(x = i) = a, \quad i = 1, \ldots 5$ and $p(x = 6) = 2a$. The sum of all probabilities must be equal to one, then:

$$a + a + a + a + a + 2a = 1$$

Therefore,

$$p(x = i) = \frac{1}{7}, \quad i = 1, \ldots 5 \quad \text{y} \quad p(x = 6) = \frac{2}{7}$$

We define the events:

- A_1 : "When the die is rolled, an even number is obtained".

- A_2 : "When the die is rolled, an odd number is obtained".

Remark

From now on, we denote B as the event "a white ball is drawn", R as the event "a red ball is drawn", and V as the event 'a green ball is drawn'.

Applying the Law of Total Probability:

$$p(R) = p(A_1)p(B/A_1) + p(A_2)p(R/A_2) = \frac{4}{7} \cdot \frac{2}{7} + \frac{3}{7} \cdot \frac{5}{14} = \frac{31}{98}$$

Excercise 15.

A box contains 10 white balls, 5 black balls, and 5 red balls. Two balls are drawn consecutively from the box. Calculate the probability that both are white in the following cases:

1. Before the second ball is drawn, the first ball is put back in the box.
2. The second ball is drawn without putting the first ball back in the box.

"

Solution. We define the event B_i: "A white ball is drawn in the i-th extraction". In both parts we are asked to calculate the value of $p(B_1 \cap B_2)$.

In the first case, events B_1 and B_2 are independent because the ball is replaced, thus:

$$p(B_1 \cap B_2) = p(B_1) \cdot p(B_2) = \frac{10}{20} \cdot \frac{10}{20} = \frac{1}{4}$$

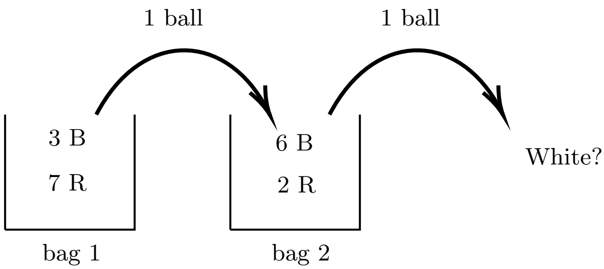
Since there is no replacement in the second case, the events B_1 and B_2 are dependent:

$$p(B_1 \cap B_2) = p(B_1) \cdot p(B_2/B_1) = \frac{10}{20} \cdot \frac{9}{19} = \frac{9}{38}$$

Excercise 16.

We have two bags. The first bag contains 3 white balls and 7 red ones, while the second bag has 6 white balls and 2 red ones. We draw a ball from the first bag and transfer it to the second bag, and then we extract a ball from the second bag. What is the probability that the ball drawn from the second bag is white? What is the probability that both balls drawn are white?

Solution.



We define the events:

- B_i : "A white ball is drawn from the i-th bag".

- R_i : "A red ball is drawn from the i-th bag".

To answer the first question, we use the Law of Total Probability:

$$p(B_2) = p(B_1) \cdot p(B_2/B_1) + p(R_1) \cdot p(B_2/R_1) = \frac{3}{10} \cdot \frac{7}{9} + \frac{7}{10} \cdot \frac{3}{9} = \frac{7}{15}$$

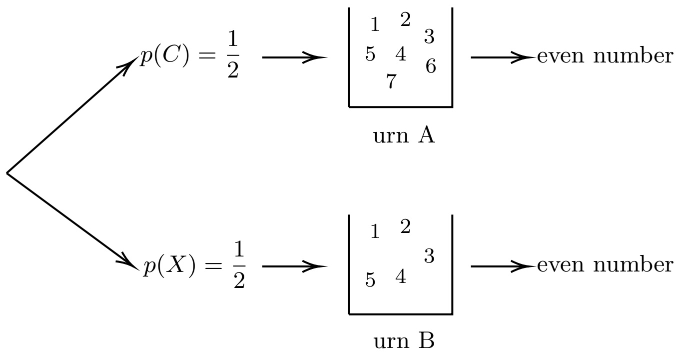
To calculate the probability of both balls being white, we apply the definition of intersection of conditional events (which are not independent):

$$p(B_1 \cap B_2) = p(B_1) \cdot p(B_2/B_1) = \frac{3}{10} \cdot \frac{7}{9} = \frac{7}{30}$$

Excercise 17.

A urn A contains 7 balls numbered from 1 to 7. In another urn B there are 5 balls numbered from 1 to 5. We toss a fair coin, and if it lands heads, we draw a ball from urn A, and if it lands tails, we draw a ball from urn B. What is the probability of getting an even number?

Solution. From now on, we will denote the event of getting "heads" as C and the event of getting "tails" as X. We represent the problem using the following diagram:



We define the events:

- A : "Extracting the number from urn A".
- B : "Extracting the number from urn B".

If it lands heads, we automatically extract a number from urn A and if it lands tails, we do it from urn B. Therefore,

$$p(A) = p(C) = \frac{1}{2} \quad \text{and} \quad p(B) = p(X) = \frac{1}{2}$$

Using the Law of Total Probability:

$p(\text{"extracting an even number"}) = p(A) \cdot p(\text{"extracting an even number"}/A)$
$+ p(B) \cdot p(\text{"extracting an even number"}/B) = \frac{1}{2} \cdot \frac{3}{7} + \frac{1}{2} \cdot \frac{2}{5} = \frac{29}{70}$

Excercise 18.

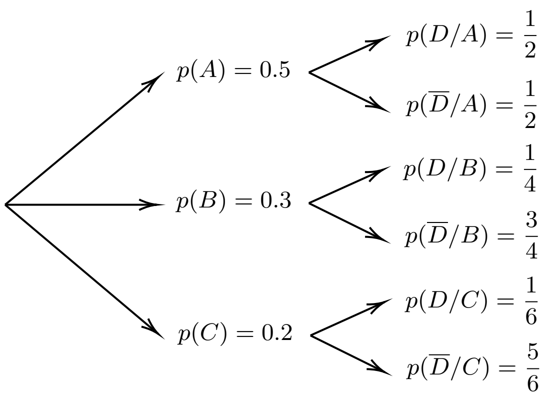
A factory has three production lines: A, B, and C. Line A produces 50% of the total cars produced, B produces 30%, and C produces the remainder. The probability that a car is defective is 1/2 for line A, 1/4 for line B, and 1/6 for line C. Find:

1. The probability that a car is defective and was produced by line A.
2. The probability that a car is defective.
3. If a car is not defective, what is the probability that it was produced by line C?

Solution. We define the events:

- A : "The car is produced in chain A".
- B : "The car is produced in chain B".
- C : "The car is produced in chain C".
- D : "The car is defective".

We represent the tree diagram:



1. We select the branch where the events A and D appear and multiply their probabilities: $p(A \cap D) = p(A) \cdot p(D/A) = 0.5 \cdot \frac{1}{2} = 0.25$.

2. We use the Total Probability Theorem. We select the branches where the event D appears:

   $p(D) = p(A) \cdot p(D/A) + p(B) \cdot p(D/B) + p(C) \cdot p(D/C) = 0,5 \cdot \dfrac{1}{2} + 0,3 \cdot \dfrac{1}{4} + 0,2 \cdot \dfrac{1}{6} = 0,358$

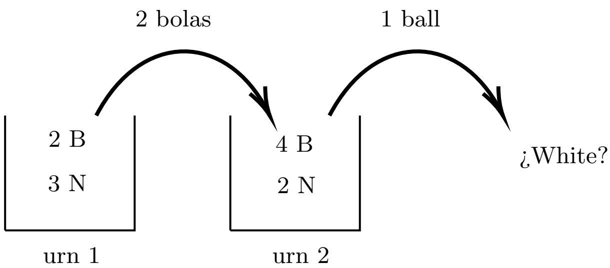
3. Applying Bayes' theorem:

$$p(C/\overline{D}) = \frac{p(C \cap \overline{D})}{p(\overline{D})} = \frac{p(C) \cdot p(\overline{D}/C)}{1 - p(D)} = \frac{0,2 \cdot \dfrac{5}{6}}{1 - 0,358} = 0,26$$

Excercise 19.

A urn contains two white balls and three black balls. A second urn has four white balls and two black balls. Two balls are transferred from the first urn to the second urn and then one ball is drawn from the second urn. What is the probability that it is white?

Solution. We represent the problem in the following diagram:



We define the events:

- A_1 : "The two balls that are transferred are white".
- A_2 : "The two balls that are transferred are black".
- A_3 : "One white and one black ball are transferred".

Since the three events are mutually exclusive and their union forms the whole sample space Ω, we can apply the Total Probability Theorem:

$$p(B) = p(A_1)p(B/A_1) + p(A_2)p(B/A_2) + p(A_3)p(B/A_3) \quad (1)$$

To calculate the values of $p(A_i)$, we cannot use the Laplace's Rule because not all events are equiprobable: there is always more probability of drawing two balls of different colors than two balls of the same color (think of the problem of the Duke of Tuscany). To use Laplace's Rule, we have to calculate as if the balls were numbered and the order mattered. Therefore:

$$p(A_1) = \frac{\text{favorable outcomes}}{\text{possible outcomes}} = \frac{V_{2,2}}{V_{5,2}} = \frac{2}{20} = \frac{1}{10}$$

$$p(A_2) = \frac{\text{favorable outcomes}}{\text{possible outcomes}} = \frac{V_{3,2}}{V_{5,2}} = \frac{6}{20} = \frac{3}{10}$$

$$p(A_3) = \frac{\text{favorable outcomes}}{\text{possible outcomes}} = \frac{V_{3,1} \cdot V_{2,1} + V_{2,1} \cdot V_{3,1}}{V_{5,2}} = \frac{12}{20} = \frac{6}{10}$$

Substituting into (1):

$$p(B) = \frac{1}{10} \cdot \frac{6}{8} + \frac{3}{10} \cdot \frac{4}{8} + \frac{6}{10} \cdot \frac{5}{8} = \frac{3}{5}$$

Excercise 20.

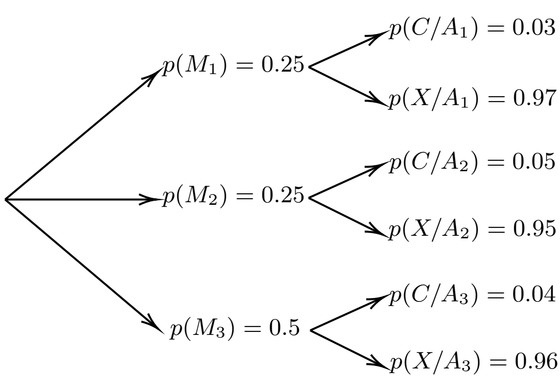
Three machines, M_1, M_2, and M_3, produce 250, 250, and 500 pieces per week, respectively. It is also known that, respectively, 3

1. The probability that it is defective.
2. If the piece is not defective, the probability that it was manufactured by machine 2.

Solution. We define the events:

- M_i : "The piece was manufactured by machine M_i".
- D : "The manufactured piece is defective".

We represent the tree diagram:



1. Applying the Total Probability Theorem, selecting the branches where the event D appears:

$$p(D) = p(M_1)p(D/M_1) + P(M_2)p(D/M_2) + p(M_3)p(D/M_3)$$

$$= 0,25 \cdot 0,04 + 0,25 \cdot 0,05 + 0,5 \cdot 0,04 = 0,04$$

2. We apply Bayes' Theorem:

$$p(M_1/\overline{D}) = \frac{p(M_1)p(\overline{D}/M_1)}{p(\overline{D})} = \frac{0,0075}{0,04} = 0,1875$$

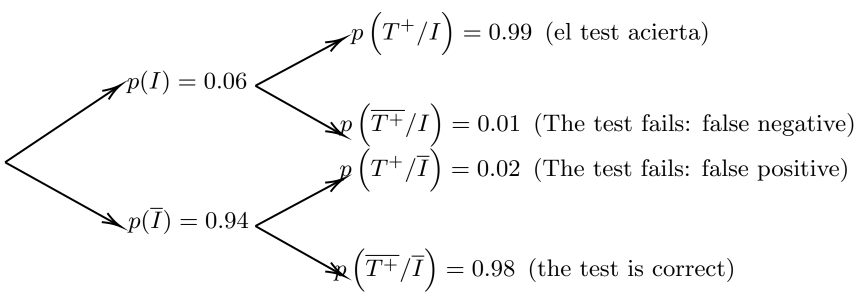
Excercise 21.

An IT security company develops an antivirus software. Before being commercialized, it needs to pass a quality control. The control concludes that if a computer is infected, which happens with a probability of 6

Solution. We define the events:

- I : "The computer is infected".
- T^+: "The test is positive".
- $\overline{T^+}$: "The test is negative".

We represent the tree diagram:



To calculate the probability we are asked for, we apply Bayes' Theorem:

$$p(I/\overline{T^+}) = \frac{p(I \cap \overline{T^+})}{p(\overline{T^+})} \quad (1)$$

First, we will calculate the probability of the event $\overline{T^+}$ occurring. To do this, we will use the Total Probability Theorem, selecting the branches of the tree where the event $\overline{T^+}$ appears:

$$p(I) \cdot p(\overline{T^+}/I) + p(\overline{I}) \cdot p(\overline{T^+}/\overline{I}) = 0,06 \cdot 0,01 + 0,94 \cdot 0,98 = 0.9218$$

Substituting into (1):

$$p(I/\overline{T^+}) = \frac{p(I \cap \overline{T^+})}{p(\overline{T^+})} = \frac{p(I) \cdot p(\overline{T^+}/I)}{p(\overline{T^+})} = \frac{0,06 \cdot 0,01}{0.9218} = 0.00065$$

Excercise 22.

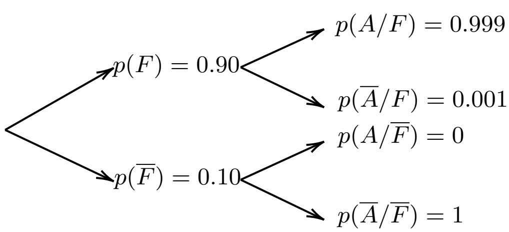
A bank has found that the probability of a customer with funds in their account writing a check with the wrong date is 0.001. On the other hand, every customer without funds writes a check with the wrong date. Additionally, it is known that 90% of the bank's customers have funds. Calculate:

1. Given a randomly chosen customer, calculate the probability that they will sign with the correct date.
2. Today, a check with the wrong date is received at the cash register. What is the probability that it is from a customer without funds?

Solution. We define the events:

- F : "The customer has funds in their account."
- A "The customer signs with the correct date."

We represent the tree diagram:



1. "To calculate the probability that a customer signs with the correct date, we apply the Total Probability Theorem:

   $$p(A) = p(F) \cdot p(A/F) + p(\overline{F}) \cdot p(A/\overline{F}) = 0,90 \cdot 0,999 + 0,10 \cdot 0 = 0,8991$$

2. We apply Bayes' Theorem:

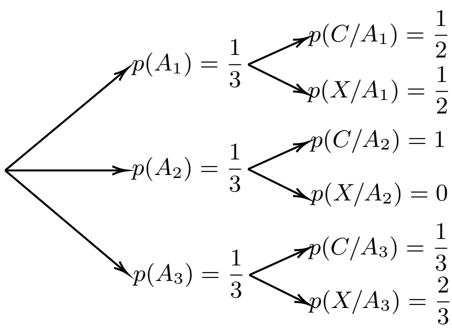
   $$p(\overline{F}/\overline{A}) = \frac{p(\overline{F} \cap \overline{A})}{p(\overline{A})} = \frac{p(\overline{F}) \cdot p(\overline{A}/\overline{F})}{1 - p(A)} = \frac{0,10 \cdot 1}{1 - 0,8991} = 0,991$$

Excercise 23.

A box contains three coins. One coin is normal, another has two heads, and the other is biased such that the probability of obtaining heads is 1/3. A coin is selected at random and flipped. Find the probability that it lands heads up. If it landed tails, what is the probability that the normal coin was selected?

Solution. We define the events and represent the diagram:

- A_1 : "The normal coin is selected."
- A_2 : "The coin with two heads is selected."
- A_3 : "The biased coin is selected."



To find the probability of obtaining a head, we have to apply the Law of Total Probability, taking into account that each of the three coins has an equal probability of being selected.

$$p(C) = \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{1}{3} = \frac{11}{18}$$

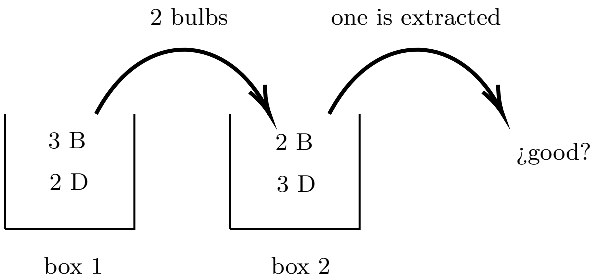
To answer the second question, we use Bayes' Theorem:

$$p(A_1/X) = \frac{p(A_1)p(X/A_1)}{p(X)} = \frac{\frac{1}{3} \cdot \frac{1}{2}}{1 - \frac{11}{18}} = \frac{9}{14}$$

Excercise 24.

A box contains three good bulbs and two defective ones, and a second box contains two good and three defective bulbs. Two bulbs are transferred from the first box to the second box, and then a bulb is drawn from the second box, which turns out to be good. What is the probability that the transferred bulbs are one good and one defective?

Solution. We represent the problem in the following diagram:



We define:

- A_1 : "Two good bulbs are transferred to the second box."
- A_2 : "Two defective bulbs are transferred to the second box."
- A_3 : "One good and one defective bulb are transferred to the second box."
- B : "The bulb is good."

To calculate the value of $p(A_3/B)$, we use the definition of conditional probability:

$$p(A_3/B) = \frac{p(A_3 \cap B)}{p(B)} = \frac{p(A_3)p(B/A_3)}{p(B)} \quad (1)$$

To calculate $p(A_1)$, $p(A_2)$ and $p(A_3)$ we will proceed in the same way as in Exercise 20 to ensure that we can apply the Laplace Rule:

$$p(A_1) = \frac{\text{favorable outcomes}}{\text{possible outcomes}} = \frac{V_{3,2}}{V_{5,2}} = \frac{6}{20} = \frac{3}{10}$$

$$p(A_2) = \frac{\text{favorable outcomes}}{\text{possible outcomes}} = \frac{V_{2,2}}{V_{5,2}} = \frac{2}{20} = \frac{1}{10}$$

$$p(A_3) = \frac{\text{favorable outcomes}}{\text{possible outcomes}} = \frac{V_{3,1} \cdot V_{2,1} + V_{2,1} \cdot V_{3,1}}{V_{5,2}} = \frac{12}{20} = \frac{6}{10}$$

The probability of the event B is calculated using the Law of Total Probability:

$$p(B) = p(A_1)p(B/A_1) + p(A_2)p(B/A_2) + p(A_3)p(B/A_3) = \frac{3}{10} \cdot \frac{4}{7} + \frac{1}{10} \cdot \frac{2}{7} + \frac{6}{10} \cdot \frac{3}{7} = \frac{16}{35}$$

If we substitute the values obtained in (1):

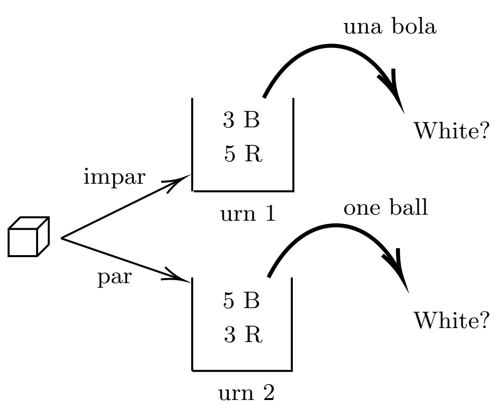
$$p(A_3/B) = \frac{p(A_3 \cap B)}{p(B)} = \frac{p(A_3)p(B/A_3)}{p(B)} = \frac{\frac{6}{10} \cdot \frac{3}{7}}{\frac{16}{35}} = \frac{9}{16}$$

Excercise 25.

A fair die is rolled. If an odd number is obtained, a ball is drawn from an urn containing 3 white and 5 red balls. If an even number appears, a ball is drawn from an urn containing 5 white and 3 red balls. Find:

1. The probability that the drawn ball is white.
2. The probability that the first urn was chosen given that a red ball was drawn in the extraction.

Solution. On the next page we see the graphical representation of our problem: the even or odd value is what leads us to one urn or another, drawing a ball from the corresponding urn.



We define the events:

- A_1 : "We extract the ball from urn 1".
- A_2 : "We extract the ball from urn 2".

If an odd number comes up, we choose the first urn, so the probability of event A_1 is the same as the probability of rolling an odd number with an unbiased die, that is, $p(A_1) = \frac{1}{2}$. The same applies to A_2.

1. To calculate the probability that the ball is white, we use the Law of Total Probability:
$$p(B) = p(A_1)p(B/A_1) + p(A_2)p(B/A_2) = \frac{1}{2}\cdot\frac{3}{8} + \frac{1}{2}\cdot\frac{5}{8} = \frac{8}{16} = \frac{1}{2}$$

2. We will use Bayes' Theorem, taking into account that the event R is the complement of event B:
$$p(A_1/R) = \frac{p(A_1)p(R/A_1)}{p(R)} = \frac{p(A_1)p(R/A_1)}{1-p(B)} = \frac{\frac{1}{2}\cdot\frac{5}{8}}{\frac{1}{2}} = \frac{5}{8}$$

Excercise 26.

An exam consists of 14 topics. One topic must be chosen from two topics selected at random. Calculate the probability that a student who has prepared 5 topics will get at least one of the studied topics.

Solution. We define event A: "The student gets at least one of the studied topics." As there are no repeated topics, all cases are equiprobable, and therefore we can apply the Rule of Laplace. Additionally, this situation allows us to calculate favorable and possible outcomes using combinations.

Instead of directly calculating the probability of the event we are asked for, it is easier to calculate the probability of its complement, $\overline{A}$, that is, the probability that none of the two randomly chosen topics have been studied. Therefore,

$$p(\overline{A}) = \frac{\binom{14-5}{2}}{\binom{14}{2}} = \frac{36}{91} \qquad \Longrightarrow \qquad p(A) = 1 - p(\overline{A}) = 1 - \frac{36}{91} = \frac{55}{91}$$

Excercise 27.

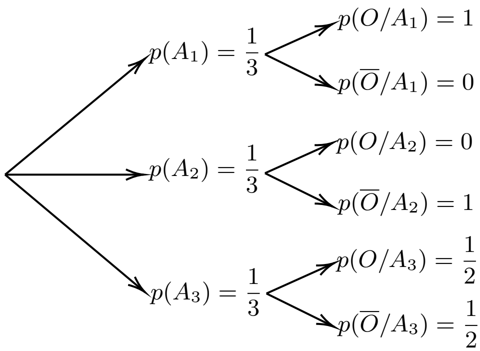
There are three bags, each containing two coins. The first bag contains gold coins, the second bag contains silver coins, and the third bag contains one gold coin and one silver coin. A bag is chosen at random and a coin is then chosen at random from that bag. If a gold coin is drawn, what is the probability that the other coin in the bag is also a gold coin?

Solution. We define the events:

- A_1 : "A coin is drawn from the first bag."
- A_2 : "A coin is drawn from the second bag."

- A_3 : "A coin is drawn from the third bag."
- O : "The drawn coin is gold."

We represent the tree diagram:



We are asked for the probability that the other coin is gold. This is equivalent to asking for the probability of having chosen the first bag, since it is the only one with two gold coins.

We use Bayes' Theorem:

$$p(A_1/O) = \frac{p(A_1)p(B/A_1)}{p(O)} \quad (1)$$

We need to find the probability of the event O using the Law of Total Probability. We select the branches of the tree where the event O occurs:

$$p(O) = p(A_1)p(O/A_1)+p(A_2)p(O/A_2)+p(A_3)p(O/A_3) = \frac{1}{3}\cdot 1+\frac{1}{3}\cdot 0+\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{2}$$

Substituting into (1):

$$p(A_1/O) = \frac{p(A_1)p(O/A_1)}{p(O)} = \frac{\frac{1}{3}\cdot 1}{\frac{1}{2}} = \frac{2}{3}$$

Excercise 28: Monty Hall problem

Let's suppose a player is in a game show in which he is faced with three doors. Behind one of them, there is a car, but behind the other two, there is nothing. He chooses one door and then the presenter, without opening the door that the player chose, opens another door behind which there is no prize. This is possible because the presenter knows where the car is. Then the presenter asks the player, "Don't you prefer to change your choice?". Would it be better to switch doors or stick with the original choice?

Solution. At first glance, it may seem like it doesn't matter whether the player changes the door or not, but it is not so because the presenter opens the other door after the player has made his choice.

It is obvious that if the player stays with his initial decision, the probability of winning is 1/3. But if he changes the door, two things can happen:

- In the first choice, he pointed to an empty door, so if he changes, the only closed door is the one with the car behind it.
- In the first choice, he pointed to the door with the car behind it. Therefore, he loses because the only closed door left is empty.

Since there are two empty doors and only one door with the car, the probability of winning if the player switches doors is 2/3.

Let's express the problem mathematically by defining the following events:

- A: "The player initially chooses an empty door."
- $\overline{A}$: "The player initially chooses the door with the car."
- G : "The player wins the contest."

Using the Law of Total Probability, we can express the probability of winning the game as:

$$p(G) = p(A)p(G/A) + p(\overline{A})p(G/\overline{A})$$

If the player switches doors, the probability of winning the car will be:

$$p(G) = p(A)p(G/A) + p(\overline{A})p(G/\overline{A}) = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 0 = \frac{2}{3}$$

If the player sticks to their initial choice and doesn't switch, the probability of winning will be:

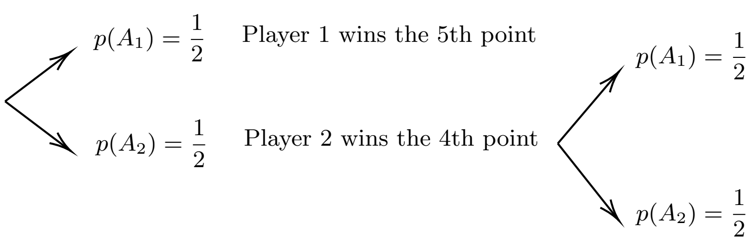
$$p(G) = p(A)p(G/A) + p(\overline{A})p(G/\overline{A}) = \frac{2}{3} \cdot 0 + \frac{1}{3} \cdot 1 = \frac{1}{3}$$

Excercise 29:

Two players bet a hundred euros on heads or tails (one chooses heads, the other tails, and whoever guesses right earns a point). The player who first reaches five points wins the bet, but the game is interrupted when the first player has four points and the second player has three. How should they split the 100 euros?

This type of problem was posed by Chevalier de Mère to his friend Blaise Pascal, which led to a lot of correspondence between him and René Descartes. This exchange of letters is considered the starting point of probability calculus.

Solution. We define the event A_i: "Player i wins one point". We will use a tree diagram:



The player 1 can win the game if after the resumption he gets it right on the first toss or misses on the first and gets it right on the second. Therefore, the probability of winning is as follows:

$$p(\text{“Player 1 wins the game”}) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$$

Player 2 would only win if after the resumption he gets it right in both tosses consecutively, so his probability of winning would be:

$$p(\text{“Player 2 wins the game”}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Player 1 would win the game with a probability of 75%, while player 2 would do so with a probability of 25%. Therefore, the fairest distribution will be to give 75 euros to the first player and 25 euros to the second.

Excercise 30.

What is more advantageous, betting on at least one six appearing in four rolls of a die or betting on at least one double six appearing when rolling two dice twenty-four times? This is another of the problems that Chevalier de Mére presented to Pascal.

Solution. Let's calculate the first probability by first obtaining the value of the probability of its opposite, that is, the probability of not getting any six in four rolls:

$$p(\text{“no six”}) = \left(\frac{5}{6}\right)^4 \quad \Longrightarrow \quad p(\text{“at least one six”}) = 1 - \left(\frac{5}{6}\right)^4 = 0.52$$

The probability of not getting any double six when rolling two dice is 35/36, since we have thirty-six possible outcomes (considering that order matters to make outcomes equiprobable and use the Laplace Rule) and thirty-five favorable. Therefore, the probability of getting at least one double six in twenty-four rolls will be:

$$p(\text{“at least one double six in 24 rolls”}) = 1 - \left(\frac{35}{36}\right)^{24} = 0.49$$

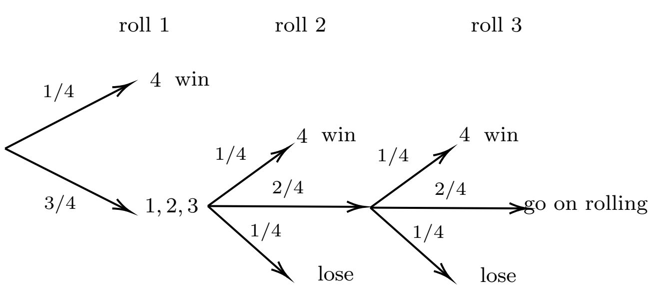
We see that it is more advantageous to make the first bet. Chevalier De Mére reasoned wrongly that, if the ratio between the number of rolls and the possible outcomes was the same (4 is to 6 as 24 to 36), the probability should also be the same.

Excercise 31.

A fair, four-sided die has faces numbered one to four. We play the following game:

We roll the die. If it lands on four, we win. If it lands on one, two, or three, we continue rolling the die. We lose if we roll the same number as the first roll, and we win if we roll a four. Find the probability of winning.

Solution. We can represent the game with the following diagram:



We define the event G_i : “Winning on the i-th roll”. Since the result of one roll does not affect the next one, we have:

- $p(G_1) = \frac{1}{4}$
- $p(G_2) = \frac{3}{4} \cdot \frac{1}{4} = \frac{1}{4} \cdot \frac{3}{4}$
- $p(G_3) = \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} = \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{2}{4}$
- $p(G_4) = \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} = \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{2}{4}$

The probability of winning will be the probability of winning on the first roll, or on the second, or on the third, etc. Furthermore, these events are mutually exclusive, therefore:

$$p(\text{“}Win'') = p(G_1 \cup G_2 \cup G_3 \ldots) = p(G_1) + p(G_2) + \ldots = \frac{1}{4} + \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{2}{4} + \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{2}{4} = \frac{1}{4}\left(1 + \frac{3}{4} + \frac{3}{4} \cdot \frac{2}{4} + \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{2}{4}\right) = \frac{1}{4}\left[1 + \frac{3}{4}\left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots\right)\right]$$
$$= \frac{1}{4}\left(1 + \frac{3}{4} \cdot 2\right) = \frac{5}{8}$$

Excercise 32.

In a Spanish deck of cards, two cards are drawn at random. Find the probability that both are cups; at least one is clubs and one is a cup and the other is a sword.

Solution. The Spanish deck of cards has 40 cards with four different suits: golds, cups, swords, and clubs. As in previous problems, we will use permutations to calculate the favorable outcomes and possible outcomes, and thus apply the Laplace Rule.

If the two cards we draw are both cups, the possible outcomes will be the variations that can be formed with ten cards taken in groups of two, while the possible outcomes will always be variations of the forty cards taken in groups of two.

$$p(\text{“two cups”}) = \frac{V_{10,2}}{V_{40,2}} = \frac{10 \cdot 9}{40 \cdot 39} = \frac{3}{52}$$

To calculate the probability of the event “at least one is a cup”, we will proceed as we have done before; we will first calculate the probability of its contrary event “none is a cup”:

$$p(\text{“none is a cup”}) = \frac{V_{30,2}}{V_{40,2}} = \frac{30 \cdot 29}{40 \cdot 39} = \frac{29}{52}$$

We took variations of thirty cards because there are thirty cards in the deck that are not cups. Now we just have to subtract the probability obtained from one:

$$p(\text{“at least one is a cup”}) = 1 - p(\text{“none is a cup”}) = 1 - \frac{29}{52} = \frac{23}{52}$$

To calculate the last probability, we first calculate the favorable outcomes:

If we take variations of twenty cards (cups and swords) in groups of two, we are including the cases where both cards are of the same suit. Then, we have to subtract the cases where both cards are of the same suit:

Variations of 20 cards taken in groups of two: $V_{20,2} = 380$

Cases where both cards are of the same suit: $V_{10,2} = 90$

Therefore, the favorable outcomes are: $380 - 90 - 90 = 200$

$$p(\text{“one is a cup and the other sword”}) = \frac{V_{20,2} - 2 \cdot V_{10,2}}{V_{40,2}} = \frac{200}{40 \cdot 39} = \frac{5}{39}$$

Excercise 33.

Let A and B be, so that $p(A) = 1/3$, $p(B) = 1/5$ and $p(A|B) + p(B|A) = 2/3$. Calculate $p(\overline{A} \cup \overline{B})$.

Solution. We will calculate the probability we are asked for using one of De Morgan's laws:

$$p(\overline{A} \cup \overline{B}) = p(\overline{A \cap B}) = 1 - p(A \cap B)$$

(1) We will use the given data to calculate the probability of $p(A \cap B)$. Applying the definition of conditional probability,

$$p(A|B) + p(B|A) = \frac{p(A \cap B)}{p(B)} + \frac{p(A \cap B)}{p(A)} = p(A \cap B)\left(\frac{1}{p(B)} + \frac{1}{p(A)}\right)$$

Substituting the data given in the problem, we have that:

$$\frac{2}{3} = p(A \cap B)\left(\frac{1}{1/3} + \frac{1}{1/5}\right) \implies p(A \cap B) = \frac{2}{24} = \frac{1}{12}$$

Subtituting into (1):

$$p(\overline{A} \cup \overline{B}) = p(\overline{A \cap B}) = 1 - p(A \cap B) = 1 - \frac{1}{12} = \frac{11}{12}$$

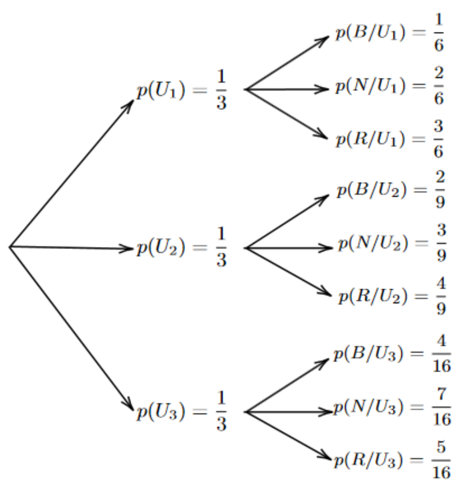
Excercise 34.

We have three urns with the following composition:

$$U_1 = \{1B, 2N, 3R\},\ U_2 = \{2B, 3N, 4R\}\ y\ U_3 = \{4B, 7N, 5R\}$$

An urn is chosen at random and a ball is drawn. What is the probability that the ball is red? If the ball turns out to be white, find the probability that it came from the third urn.

Solution. We define the event U_i : "A ball is drawn from the i-th urn". We represent the tree diagram:



To calculate the probability of drawing a red ball, we apply the Law of Total Probability, by summing the products of the probabilities of the branches where the event R appears:

$p(R) = p(U_1)p(R/U_1)+p(U_2)p(R/U_2)+p(U_3)p(R/U_3) = \frac{1}{3}\cdot\frac{3}{6}+\frac{1}{3}\cdot\frac{4}{9}+\frac{1}{3}\cdot\frac{5}{16} = 0.419$

We now need to use Bayes' Theorem to calculate the probability of the event U_3/B:

$$p(U_3/B) = \frac{p(U_3)p(B/U_3)}{p(B)}$$

The first thing we need to do is calculate the probability of the event B using the Law of Total Probability:

$p(B) = p(U_1)p(B/U_1)+p(U_2)p(B/U_2)+p(U_3)p(B/U_3) = \frac{1}{3}\cdot\frac{1}{6}+\frac{1}{3}\cdot\frac{2}{9}+\frac{1}{3}\cdot\frac{4}{16} = 0.213$

therefore,

$$p(U_3/B) = \frac{p(U_3)p(B/U_3)}{p(B)} = \frac{\frac{1}{3}\cdot\frac{4}{16}}{0.213} = 0.391$$

Excercise 35.

The probability that a certain system has n failures is given by the following formula:

$$p_n = \frac{1}{e}\cdot\frac{1}{n!} \quad n = 0, 1, \ldots$$

If n failures occur, the system stops working with probability $1 - (1/2)^n$. Calculate the probability that the system had n failures if it has stopped working.

Solution. Let's consider the event D: "The system stops working". We apply Bayes' Theorem:

$$p(\text{“}n\text{ failures”}/D) = \frac{p(\text{“}n\text{ failures”})\cdot p(D/\text{“}n\text{ failures”})}{\sum_{k=0}^{\infty} p(\text{“}k\text{ failures”})\cdot p(D/\text{“}k\text{ failures”})}$$

$$= \frac{\big(1-(1/2)^n\big)\cdot\frac{1}{e}\cdot\frac{1}{n!}}{\sum_{k=0}^{\infty}\Big(1-(1/2)^k\Big)\cdot\frac{1}{e}\cdot\frac{1}{k!}}$$

Now, we divide the numerator and denominator by $1/e$ and expand the infinite series that appears in the numerator:

$$\sum_{k=0}^{\infty}\Big(1-(1/2)^k\Big)\cdot\frac{1}{k!} = 1 - \sum_{k=0}^{\infty}(1/2)^k\cdot\frac{1}{k!} = 1 - e^{1/2}$$

then:

$$p(\text{“n fallos”}/D) = \frac{\big(1-(1/2)^n\big)\cdot\frac{1}{n!}}{1-e^{1/2}}$$

Excercise 36.

We have two Spanish decks of cards. One of them is missing a card and we do not know which one. We choose a deck at random and draw a card, also randomly. Calculate the probability that it is a gold card.

Solution. We define the following events:

- B_1 : "The complete deck is selected".
- B_2 : "The deck missing a card is selected".
- O : "A gold card is drawn".

We can use the Law of Total Probability based on the events B_1 and B_2:

$$p(O) = p(B_1)p(O/B_1) + p(B_2)p(O/B_2)$$

Since we choose either of the two decks with equal probability, it is clear that

$$p(B_1) = p(B_2) = \frac{1}{2}$$

Calculating the probability of event O/B_1 is also straightforward since we have selected the complete deck:

$$p(O/B_1) = \frac{10}{40} = \frac{1}{4}$$

However, the calculation of $p(O/B_2)$ is not immediate because we know that a card is missing, but we do not know which suit it belongs to. To simplify the notation, let us denote $A = O/B_2$ and define:

- C_1 : "A card of the suit "oros" is missing".
- C_2 : "A card of the suit "copas" is missing".
- C_3 : "A card of the suit "espadas" is missing".
- C_4 : "A card of the suit "bastos" is missing".

"

$$p(A) = p(A)p(C_1/A) + p(A)p(C_2/A) + p(A)p(C_3/A) + p(A)p(C_4/A) =$$

$$= \frac{1}{4} \cdot \frac{9}{39} + \frac{1}{4} \cdot \frac{10}{39} + \frac{1}{4} \cdot \frac{10}{39} + \frac{1}{4} \cdot \frac{10}{39} = \frac{1}{4}$$

Excercise 37.

We have three urns: one with three white balls, one with one white and two black balls, and one with one white and two red balls. We extract one ball from each urn without replacement, and then we perform another extraction without replacement. We know that in this second extraction, the balls are of different colors. What is the probability that in the first extraction, the three balls were of the same color?

Solution. We consider the following events:

- B_{ij}: "A white ball was drawn from urn i in the j-th extraction".
- N_{ij}: "A black ball was drawn from urn i in the j-th extraction".
- R_{ij}: "A red ball was drawn from urn i in the j-th extraction".

If the balls extracted in the first step are of the same color, they must necessarily be white, that is, the event $B_{11} \cap N_{21} \cap R_{31}$ has occurred.

On the other hand, for the second extraction to yield balls of different colors, a white, a black, and a red ball must have been drawn, that is, the event $B_{12} \cap N_{22} \cap R_{32}$ has occurred.

Therefore, the probability we are asked to calculate is:

$$p[(B_{11} \cap N_{21} \cap R_{31})/(B_{12} \cap N_{22} \cap R_{32})] \quad (1)$$

The events B_{11}, N_{21}, and R_{31} are clearly independent, so we can multiply their probabilities:

$p[(B_{11}/(B_{12} \cap N_{22} \cap R_{32})] = p(B_{11}/B_{12}) = 1$

$p[N_{21}/(B_{12} \cap N_{22} \cap R_{32})] = p(N_{21}/N_{22}) = p\left(\dfrac{N_{21} \cap N_{22}}{N_{22}}\right) = \dfrac{\dfrac{2}{3} \cdot \dfrac{1}{2}}{\dfrac{2}{3} \cdot \dfrac{1}{2} + \dfrac{1}{3} \cdot 1} = \dfrac{1}{2}$

$p[(R_{31})/(B_{12} \cap N_{22} \cap R_{32})] = p(R_{31}/R_{32}) = p\left(\dfrac{R_{31} \cap R_{32}}{R_{32}}\right) = \dfrac{\dfrac{2}{3} \cdot \dfrac{1}{2}}{\dfrac{2}{3} \cdot \dfrac{1}{2} + \dfrac{1}{3} \cdot 1} =$

$\dfrac{1}{2}$

Subtituting into (1):

$$p[(B_{11} \cap N_{21} \cap R_{31})/(B_{12} \cap N_{22} \cap R_{32})] = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

A competitive exam consists of 20 topics. In a drum, there are 20 numbered balls from 1 to 20. Two balls are drawn at random, and one of the two topics that have come out is chosen. If a student has prepared 6 topics, calculate the probability that at least one of the chosen topics is one of those studied. Also, calculate the minimum number of topics the student must prepare to have a probability greater than 90

Solución.

We define A : "At least one of the studied topics is chosen". Then, it is easier to first calculate the opposite event of A, that is, $\overline{A}$: "None of the studied topics are chosen".

$$p(A) = 1 - p(\overline{A}) = 1 - \frac{\binom{20-6}{2}}{\binom{20}{2}} = 1 - \frac{91}{190} = \frac{99}{190} = 52.1\%$$

Let's call now n the number of topics prepared by the candidate. Then:

$$p(A) = 1 - p(\overline{A}) = 1 - \frac{\binom{20-n}{2}}{\binom{20}{2}} > 0.9 \quad \Longrightarrow \quad 1 - \frac{(20-n)(19-n)}{20 \cdot 19} > 0.9$$

Operating, we will get the inequality:

$$\frac{39n}{380} - \frac{n^2}{380} > 0.9 \quad \longleftrightarrow \quad n^2 - 39n + 342 > 0$$

We can conclude that the student must study at least 14 topics.

Excercise 38.

A breathalyzer test conducted by the Police has an accuracy of 0.8, meaning it tests positive when the maximum limit is exceeded and negative when it does not exceed the minimum limit. Since this probability is not very high, drivers suspected of driving under the influence are subjected to a second, more precise test. The new test has 100% reliability with a sober driver, while it has a 10% error rate with drunk drivers. Both tests can be assumed to be independent. It is also known that the probability of a driver stopped by the police driving under the influence is 0.05. Calculate:

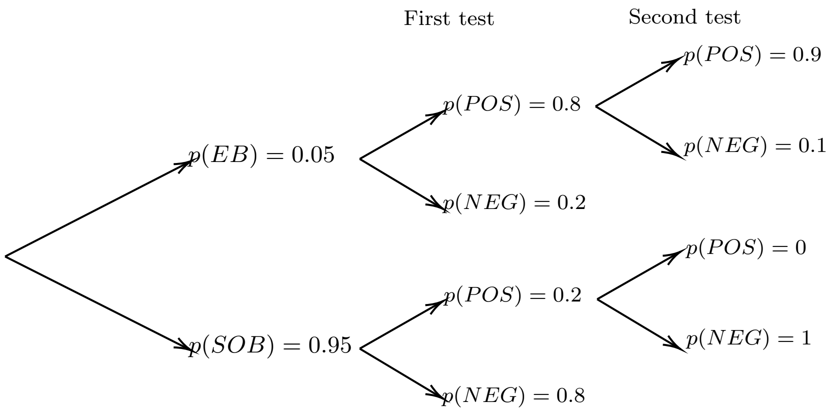
1. Proportion of drivers stopped by the police who will not be subjected to a second test.

2. Proportion of drivers stopped by the police who will be subjected to a second test and will not be fined.

3. Probability that the driver from the previous section actually had a blood alcohol content higher than the legal limit at the time of being stopped by the police. What proportion of drunk drivers will avoid a fine?

4. What proportion of drunk drivers will avoid a fine?

Solution. We represent the problem in the following tree diagram. We have omitted the conditioned events to simplify the notation:



In the first section, they ask us for the probability that a driver tests negative on the first test:

p("1^{er} negative test") $= p(EB) \cdot p(NEG) + P(SOB) \cdot p(NEG) = 0,05.0,2 + 0,95.0.8 = 0.761$

There is a 76.1% of drivers who will not be subjected to a second test.

In the second test, the drivers who tested positive in the first test are included. We are now asked about those who will test negative:

p("2^o negative test") $= p(EB) \cdot p(POS) \cdot P(NEG) + P(SOB) \cdot p(POS) \cdot p(NEG) = 0,05.0,8.0,1 + 0,95.0,2.1 = 0.194 \quad (19.4\%)$.

Finally, we are asked for the proportion of drunk drivers who passed the second test, that is, those who were drunk, tested positive in the first test, but tested negative in the second. $p(EB/$ "2^o negative test"$) = \dfrac{p(EB \cap \text{"}2^o \text{ negative test"})}{p(\text{"}2^o \text{ negative test "})} = \dfrac{0,05.0,8.0,1}{0,194} = 0.02$

The proportion will be 2%. These are drivers who tested positive in the first test but negative in the second.

To answer the last question, we need to add the drunk drivers who test negative on the first test:

$$p(EB\ \cap\ \text{“}1^{er}\text{ negative test”}) = 0,05 \cdot 0,2 = 0,01$$

Therefore, in total, there will be a 3% of drunk drivers who will avoid the fine.